Pre RMO 2014 Answers, Hints, Solutions, Analysis/Comments

Set –A

Q.1 A natural number k is such that $k^2 < 2014 < (k+1)^2$. What is the largest prime factor of k?

Ans: $\mathbf{11}$

Hint: Easy Question

Solution: $44^2 < 2014 < 45^2 \Rightarrow k = 44 = 11 \cdot 2 \cdot 2.$

Analysis/Comment: One could define a function giving the integer part of a number in its decimal representation. The standard notation for such function is $[x]$. Lots of its properties can be deduced just based on its understanding. It is suggested to the reader therefore to deduce the same. In this notation, we have $k = [\sqrt{2014}] = 44$.

Q.2 The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014^{th} term of the sequence?

Ans: $\mathbf{370}$

Hint: Find the first few terms.

Solution: The second term is 73, third term being 370, fourth term being 370 itself. Meaning that from third term onwards, all terms are same giving the 2014^{th} term as 370.

Analysis/Comment: The number 370 is the sum of the cubes of its digits. The whole numbers which are the sum of the cubes of its digits are called Armstrong Numbers. There are only 6 Armstrong numbers namely $0, 1, 153, 370, 370$ and 407. Let us call the numbers which are the sum of n^{th} powers of its digits as Armstrong Numbers of order n. The reader is invited to prove that there are only finite number of Armstrong Numbers of order n although finding them may be a tough task without the aid of computers.

Q.3 Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20, BC = 70$ and $CD = 90$, then what is the value of DA?

Ans: $\mathbf{60}$

Hint: Use the Pythagoras's theorem repeatedly to get the required length.

Solution: Suppose E is the point of intersection of the diagonals, thus $AB^2 + CD^2 = (AE^2 + BE^2) + (CE^2 + DE^2) = (BE^2 + CE^2) + (DE^2 + AE^2) = BC^2 + AD^2 \Rightarrow AD^2 = 20^2 + 90^2 - 70^2 = 60^2 \Rightarrow AD = 60.$

Analysis/Comment: One could find the required length by chasing it, i.e. assuming one length as x and finding the rest in terms of it therefore either getting the value of x or getting the answer independent of x. Such techniques are quiet useful in problems like this one in case it doesn't strike what to start with. Letting $AE = x \Rightarrow BE = \sqrt{70^2 - x^2} \Rightarrow CE = \sqrt{70^2 + x^2 - 20^2} \Rightarrow DE = \sqrt{90^2 + 20^2 - 70^2 - x^2} \Rightarrow AD = \sqrt{90^2 + 20^2 - 70^2}$. Note that letting the length as x, simplifies the notations. In a problem with lots of lengths, it is a good practice to denote lengths by small single letters as against sticking to something like AE. In a problem of finding angles, one can similarly adopt the *Angle Chasing* approach. Also observe that the calculations are done only at the last step. It is also a good practice to do the calculations at the end as it reduces one's chances of making a numerical mistakes in-between. Even the calculations in the last step can be effectively carried out by writing $90^2 + 20^2 - 70^2 = 90^2 - 50 \cdot 90 = 90 \cdot (40) = 36 \cdot 100$. It may not look advantageous in a problem like this one, but at other places this would pay off. Such practice keeps the essence of the problem intact and helps reaching the desired goal with a less effort.

Q.4 In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle?

Ans: 49

Hint: Use triangle inequality to deduce the sides first and then the perimeter.

Solution: Denote one side by x, thus another side would be $3x$. Triangle inequality gives $x + 3x > 17, x + 17 > 3x, 3x + 17 > x$. Combining this, we get $\frac{17}{2} > x > \frac{17}{4}$. Thus giving possible integer values of x as $5, 6, 7, 8$, hence giving the maximum perimeter as 49.

Analysis/Comment: Triangle inequality is one of the basic tools to analyse a triangle when hardly any data us available to us about the triangle. The inequality is a two way implication, i.e. for any triangle, the triangle inequality holds and for any three positive numbers, if the three inequalities hold then they can form a triangle. Note that the inequalities are strict. If equality occur in any one of the three, then the triangle would degenerate to a segment containing the three points representing the vertices of the triangle. Using triangle inequality wisely one can deduce that in a triangle, $1 \leq \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$ **(JEE)**. You may try to justify it.

Q.5 If real numbers a, b, c, d, e satisfy

$$a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3,$$

what is the value of $a^2 + b^2 + c^2 + d^2 + e^2$?

Ans: 10

Hint: A simple question

Solution: $b = a - 1, c = a - 2, d = a - 3, e = a - 4$. $a + b + c + d + e = a - 2 \Rightarrow a = 2 \Rightarrow b = 1, c = 0, d = -1, e = -2$. Thus $a^2 + b^2 + c^2 + d^2 + e^2 = 10$.

Analysis/Comment: Such questions are bonus in a paper like this. Lookout for the same.

Q.6 What is the smallest possible natural number n for which the equation $x^2 - nx + 2014 = 0$ has integer roots?

Ans: **91**

Hint: If the roots are integers, then they are the divisors of 2014.

Solution: If the roots are α, β, then we have $\alpha\beta = 2014$. As they are integers, α, β must be divisors of $2014 = 2 \cdot 19 \cdot 53$, giving $\{\alpha, \beta\} = \{1,2014\}$ *or* $\{-1, -2014\}$ *or* $\{2, 1007\}$

or $\{-2, -1007\}$ *or* $\{19,106\}$ *or* $\{-19, -106\}$ *or* $\{38,53\}$ *or* $\{-38, -53\}$. Also $n = \alpha + \beta$, has the least positive value of n as $38 + 53 = 91$.

Analysis/Comment: In a quadratic equation with leading coefficient one, if the roots are integers, then they are the divisors of the constant term. (Here we're including negative divisors.) This can be used to our advantage a lot number of times. A couple of questions last year were based on this. The solution can be made more effective if one recognises that the roots must be positive hence eliminating half the cases above. Further if one observes that the sum of two integers whose product is given is least when their difference is least. This can be justified from the observation $(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$. This will at once gives us the required pair of roots without even having to list them. Infact one can invert the problem and give the sum of two natural numbers and ask to maximize the product. Even here, the product is maximum when the difference is least, i.e. either zero or one. This observation can be extended to set of m natural numbers, i.e. if the sum of m natural numbers is given, then their product is maximum when the numbers are all almost equal. Here the word almost equal means that the required numbers differ from one another by atmost one. The reader is invited to prove this as well as to give an algorithm to find these numbers in general. To give an example, suppose the sum of seven integers is 100, then their product is maximum when five of them are 14 and two are 15. It is to be noted that however, it may be difficult to solve this question when the product of the numbers is given and the sum is to be minimised in general. One may just have to rely on brute force approach of finding all combinations and hence minimise the sum.

Q.7 If $x^{(x^4)} = 4$, what is the value of $x^{(x^2)} + x^{(x^8)}$?

Ans: **258**

Hint: Make a guess for x.

Solution: $x = \sqrt{2} \Rightarrow x^2 = 2, x^4 = 4, x^8 = 16 \Rightarrow x^{(x^2)} + x^{(x^8)} = 2 + 256 = 258$.

Analysis/Comment: It may not be obvious to guess the exact value, but we can start by assuming it to be some power of 2 and then guessing it right as $x = \sqrt{2}$. This will give the required answer and in a paper like this no justification is required. But an intriguing student must also lookout for a justification that there is no other solution than $\sqrt{2}$. Infact in this question this is not the case. Clearly $x = \sqrt{2}$ satisfies the equation, but note that if $x = x_0 > 0$ is a solution, then $x = -x_0$ may also be a solution provided the left side is defined at $-x_0$, meaning $x = -\sqrt{2}$ is another solution. This solution however, still gives the same value of the required expression as 258, so nothing is lost. Let us now prove that these two are the only solutions. We'll prove no other positive solution exists. Observe $x = 1$ or 0 does not satisfy the equation, so either $0 < x < 1$ or $1 < x < \sqrt{2}$ or $\sqrt{2} < x$. These cases imply respectively that $0 < x^{(x^4)} < 1$ or $1 < x^{(x^4)} < 4$ or $4 < x^{(x^4)}$. Each is a contradiction as $x^{(x^4)} = 4$. Hence there is only one positive solution namely $\sqrt{2}$. We can prove on similar lines that no other negative solution exists. Note that this needs to be proven separately as it may happen that there is a negative solution $x = -x_0$, but $x = x_0$ may or may not be a solution. Now although this justification may not seem to be justifying the limited time one is given for the problem, it is to be noted that the justification may not have been expected at all. All that is needed is the ability to make a good guess. Because many a times, finding a solution may be more difficult than proving its uniqueness. Whatever is done here to prove the uniqueness of the solution is to be taken as a standard practice once we observe the solution and this method of inequalities may be applicable elsewhere.

Q.8 Let S be a set of real numbers with mean M. If the means of the sets $S \cup \{15\}$ and $S \cup \{15,1\}$ are $M + 2$ and $M + 1$, respectively, then how many elements does S have?

Ans: **4**

Hint: Get two equations in two variables.

Solution: Suppose the set has n elements, then we have $(n + 1)(M + 2) = nM + 15, (n + 2)(M + 1) = nM + 16$. Solving, we get $n = 4, M = 5$.

Analysis/Comment: An easy question once we set up the variables and deduce the equations. One should note that in this question we had assumed that the set S does not contain the elements 15 and 1. Although it may seem obvious, in a subjective paper, one needs to justify the same. It is not difficult to prove it however, as otherwise the set $S \cup \{15\}$ would be same as S and thus would have same mean as M which is not possible. Similarly one could prove that 1 is not the element of S. In a subjective paper, one should not leave behind such loopholes in their solutions as there is a chance to miss a genuine solution altogether. As an example, Q.7 had two solutions $\sqrt{2}$ and $-\sqrt{2}$ for x, the second of which is likely to have been missed under the innocent presumption that the bases are taken to be positive. One must understand that the bases in an exponential function $f(x) = a^x$ are taken to be positive so that it is easy to define domain. One can still define an exponential function for a negative base, but then the domain in that case would become highly discretized (In particular a proper subset of the set of rational numbers). By the word highly discretized, I mean between any two numbers in the domain, we can always find a number not in the domain and between any two numbers not in the domain, we can always find a number in the domain, just like the set of rationals.

Q.9 Natural numbers k, l, p and q are such that if a and b are roots of $x^2 - kx + l = 0$ then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of $x^2 - px + q = 0$. What is the sum of all possible values of q?

Ans: 4

Hint: q is the product of the roots and it is a natural number.

Solution: $q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 2 + \frac{1}{ab} = l + 2 + \frac{1}{l}$. As q and l are natural numbers, we must have $\frac{1}{l} = q - l - 2$ a natural number. Hence $l = 1$. Thus giving $q = 4$.

Analysis/Comment: The problem is straight forward once we write q in terms of l and identify $\frac{1}{l}$ being a natural number. The solution does not require k and p to be integers. The data is just superfluous.

Q.10 In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX: XB = 1: 2$ and $AY: YC = 2: 1$. If the area of triangle AXY is 10 then what is the area of triangle ABC?

Ans: 45

Hint: Join B and Y and use the result that the areas of triangle with same height are in the same ratio as their bases.

Solution: $\frac{A(\Delta AXY)}{A(\Delta BXY)} = \frac{1}{2} \Rightarrow A(\Delta BXY) = 20 \Rightarrow A(\Delta ABY) = 30$. Now $\frac{A(\Delta ABY)}{A(\Delta CBY)} = \frac{2}{1} \Rightarrow A(\Delta CBY) = 15 \Rightarrow A(\Delta ABC) = 45$.

Analysis/Comment: A Good problem based on the result about areas of triangles with same height. A very similar problem appeared last year (Q.19 Set A).

Q.11 For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \le y$ satisfy the equation $xy = x + y + (x, y)$?

Ans: 3

Hint: Prove x cannot be more than 3.

Solution:

For $x = y$, we get $(x, y) = (3,3)$.

For $x < y$, suppose $3 \le x \Rightarrow 3y \le xy = x + y + (x, y) < y + y + y = 3y$, which is contradiction. Hence either $x = 2$ or 1. $x = 1$ does not give a solution. $x = 2 \Rightarrow y = 2 + (2, y)$. Since $(2, y)$ is either 1 or 2, we get the solutions as $(x, y) = (2,3)$ or $(2,4)$. Hence there are total three solutions.

Analysis/Comment: This is one of the typical questions in number theory that eliminates the possibility of one variable beyond a certain number using inequalities, in this case 3. It is quite wonderful how inequalities are useful in solving equations. Many similar questions may be solved using this technique. Here we were seeking solutions which are integers. An equation which is restricted to set of integers is called a Diophantine Equation. Number theory is full of such Diophantine Equations. In fact Number theory essentially deals with integers hence it has flourished under many problems requiring solutions as integers. One of the important Diophantine Equations is the equation $x^2 + y^2 = z^2$ whose solutions are called the Pythagorean Triplets. There are infinite of these and are given by the 3 parameter solution $\left(2kab, k(a^2 - b^2), k(a^2 + b^2)\right)$ for integers a, b, k. It is astonishing that a solution to the generalized version of this equation $x^n + y^n = z^n$ simply does not exist for any integer $n > 2$. Readers here are warned against trying to prove the same!

Q.12 Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BDC = 90^0$. Let the incircles of triangles ABD and BCD touch BD at P and Q, respectively, with P lying in between B and Q. If $AD = 999$ and $PQ = 200$ then what is the sum of the radii of the incircles of triangles ABD and BDC?

Ans: **799**

Hint: Length Chasing.

Solution: Assume that the incircle of ABD touches AD at E and the inradius of ABD and BCD be r_1 and r_2. Thus $AE = r_1 \Rightarrow ED = 999 - r_1 \Rightarrow DP = DE = 999 - r_1 \Rightarrow r_2 = DQ = DP - PQ = 799 - r_1 \Rightarrow r_1 + r_2 = 799$.

Analysis/Comment: This one is also a good example of a *Length Chasing* problem like Q.3. Some practice with standard results for lengths associated with incircle (excircles) configuration in a triangle gives you an easy ride. The Incircle/Excircle configuration, The Circumcircle configuration in a triangle, gives rise to a number of intriguing problems and results in a triangle, needless to mention, couple of proofs of Heron's Formula, a proof of Euler line, the distances between Circumcentre-Incentre/Excentres, lengths of angle bisectors, the ultimate Feuerbach's theorem (With compass and pencil, try drawing the incircle-excircles and then a circle that passes through the midpoints of the sides of the triangle. You'll be surprised!). It is highly recommended to get through an exercise of thoroughly analysing these configurations and get a mastery of the same.

Q.13 For how many natural numbers n between 1 and 2014 (both inclusive) is $\frac{8n}{9999-n}$ an integer?

Ans: **1**

Hint: Assume $\frac{8n}{9999-n} = k$, rewrite n in terms of k and use conditions on n.

Solution: $1 \leq n = \frac{9999k}{8+k} \leq 2014 \Rightarrow \frac{8}{9998} \leq k \leq \frac{16112}{7985}$ as $8 + k > 0$. This gives two integer values of k as 1 and 2. For $k = 2, n$ is not an integer, hence only one value is possible.

Analysis/Comment: This is pretty straight forward question once we convert it into the variable k. Note that the condition we got for k is weak, i.e. a necessary but not a sufficient condition. In simpler terms, we eliminated $k = 2$ although it is an integer as the value of n for it is not. One needs to be alert in such cases as there are always possibilities of getting extraneous answers. Common places where one gets extraneous answers is when we square, or multiply both sides by some quantity, or use an identity which is not true for a few values or when we differentiate a differential equation etc. A good example is finding x such that $\sin x + \cos x = 1$. One can get a simpler equation by squaring this equation to get $\sin 2x = 0$, whose roots are $x = n\pi$. However not all are the roots of the original equation. Getting extra answers this way is a very common mistake and this can be avoided by checking the set of solution we have got into the original equation and thus eliminating the extra ones. On the other hand, sometimes some genuine solution may be lost which is also a very common mistake. As a simple example, finding the value of x from $\sin 2x = 2\cos x$. Most of us end up with $\sin x = 1$, whereas a genuine solution $\cos x = 0$ is missed altogether. Common places where a genuine solution may be missed is while cancelling a common factor from both sides, while taking square roots (eventh root in general) etc.

Q.14 One morning, each member of Manjul's family drank an $8-$ ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank $\frac{1}{7}^{th}$ of the total amount of milk and $\frac{2}{17}^{th}$ of the total amount of coffee. How many people are there in Manjul's family?

Ans: 8

Hint: The ratio of total Milk+Cofee to that drank by Manjul must be an integer.

Solution: Assume that the total milk is $7M$ ounces and total coffee is $17C$ ounces. The question requires that $\frac{7M+17C}{M+2C}$ be an integer (which equals the number of people in the house). Now $\frac{7M+17C}{M+2C} = 7 + \frac{3C}{M+2C}$. Also $0 < \frac{3C}{M+2C} < \frac{3}{2} \Rightarrow \frac{3C}{M+2C} = 1$ as it needs to be an integer. Hence there are 8 people in the house.

Analysis/Comment: This is one of the difficult but beautiful problems in the paper. I can't help but compare the problem with this classic.

On one fine morning, a hunter started from his home for hunting. He went one mile south and one mile west and hunted a bear. Then he went one mile north just to find himself back at his home. What colour has the bear?

Again you're invited to explore this problem. There just doesn't seem an apparent connection of what is given and what is asked. But then that's the beauty of the problem. Note that one could've started with M ounces of Milk and C ounces of coffee. But that would unnecessarily make a few calculations tedious and make the problem loose its lust. The mere starting point makes the problem more readable and natural, all one needs is the right variables at right place. Observe how the term $\frac{7M+17C}{M+2C}$ automatically invites you to reduce it to a lower proper fraction. Carrying denominators all over the places would not be a good idea. It is always therefore recommended to make/use substitutions that eliminate denominators. Also note that the amount $8-$ ounce in the question is also not required anywhere. It can be just any amount as long as everyone drinks equal amounts.

Q.15 Let XOY be a triangle with $\angle XOY = 90^0$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY?

Ans: **26**

Hint: Pythagoras's Theorem.

Solution: Using Pythagoras's theorem for ΔMOY and ΔNOX, we get $YM^2 = YO^2 + \frac{OX^2}{4}$ and $XN^2 = \frac{YO^2}{4} + OX^2$. Adding the two equations, one gets $YM^2 + XN^2 = \frac{5}{4}(YO^2 + +XO^2) = \frac{5}{4}XY^2 = 22^2 + 19^2 \Rightarrow XY = 26$.

Analysis/Comment: A straight application of Pythagoras's theorem. Here YM and XN are medians of the triangle. In some other variation of this problem, using Apollonius's theorem may be useful as it directly relates the length of a median with the sides. One should be therefore free to exercise these alternatives.

Q.16 In a triangle ABC, let I denote the incenter. Let the lines AI, BI and CI intersect the incircle at P, Q and R, respectively. If $\angle BAC = 40^0$. what is the value of $\angle QPR$ in degrees?

Ans: **55**

Hint: Draw a neat diagram and chase the angles.

Solution: Suppose I is the incentre of the triangle, then $\angle RIP = 180 - \frac{A+C}{2}$ (Why?) $\Rightarrow \angle IRP = \angle IPR = \frac{A+C}{4}$. Similarly $\angle IPQ = \frac{A+B}{4} \Rightarrow \angle QPR = \frac{2A+B+C}{4} = 55^0$.

Analysis/Comment: It should be noted that the points P, Q, R are not the points where the incircle touches the sides. It is therefore challenging to draw a neat diagram so as to keep these points away from the points of contact that too by keeping the lines AP, BQ, CR concurrent at I. Even then I may not look much like the centre of the incircle. But once this's achieved, you realize it's just another Angle Chasing, i.e. find as many obvious angles as possible and then get the desired one. As already pointed out earlier, it is a good idea to be ready with such configurations where lots of new problems can be made.

Q.17 For a natural number b, let $N(b)$ denote the number of natural numbers a for which the equation $x^2 + ax + b = 0$ has integer roots. What is the smallest value of b for which $N(b) = 6$?

Ans: **60**

Hint: Get a relation for $N(b)$ and then proceed.

Solution: Clearly as in Q.6, if the roots are integers, then they must be divisors of b, in this question, the negative divisors (As the sum of the roots is negative). Therefore the question reduces to finding different unordered pairs of negative divisors of b whose product is b. Now if b is a perfect square, there are odd number of divisors and if it is not, then there are even number of divisors, therefore in the former case, $N(b) = \frac{d+1}{2}$ whereas in the later it is $\frac{d}{2}$, where d is the number of divisors of b. Therefore we need to find the least number with either 11 or 12 divisors. The least number with 11 divisors is 2^{10}, whereas the candidates for 12 divisors are $2^{11}, 2^5 \cdot 3, 2^3 \cdot 3^2, 2^2 \cdot 3 \cdot 5$ the least of which is 60.

Analysis/Comment: The question requires two things, first is to identify the expression for $N(b)$ and the second is to find the least number with given number of divisors. The number of divisors of a number whose prime factorization is $2^{n_1}3^{n_2}5^{n_3}$ ... is $(n_1+1)(n_2+1)(n_3+1)$ ... .It then just requires you to identify the least of such combinations. Many questions regarding the number of divisors of a number can be framed such as how many odd/even/multiples of 3/5/perfect square/cube divisors of a number are there, what is their product/sum/sum of squares/reciprocals etc. You may investigate these cases.

Q.18 Let f be a one-to-one function from the set of natural numbers to itself such that $f(mn) = f(m)f(n)$ for all natural numbers m and n. What is the least possible value of $f(999)$?

Ans: 24

Hint: Have a hunch for the answer and follow the intuitions.

Solution: Put $m = 1 \Rightarrow f(n) = f(1)f(n) \Rightarrow f(1) = 1$ as $f(n) \neq 0$. As $f(x)$ is a one-one function $\Rightarrow f(n) > 1\ \forall n > 1$. Now $f(999) = f(3)^3 f(37) \geq 2^3 \cdot 3 = 24$. We claim that equality does occur at 24 by setting $f(3) = 2, f(2) = 37, f(37) = 3, f(p) = p\ \forall$ primes $p \neq 2,3,37$ and $f(x)$ to be obtained from $f(mn) = f(m)f(n)$, whenever x is not prime. $f(999)$ will obtain its least value for this function once it is justified that the function is one-one. Now assume $f(N) = f(M)$ and suppose the prime factorization of N and M be $2^{x_1}3^{x_2}5^{x_3} \ldots 37^{x_{12}} \ldots$ and $2^{y_1}3^{y_2}5^{y_3} \ldots 37^{y_{12}} \ldots$

$$\Rightarrow f(N) = f(2^{x_1}3^{x_2} \ldots 37^{x_{12}} \ldots) = f(2)^{x_1} f(3)^{x_2} \ldots f(37)^{x_{12}} \ldots = 37^{x_1}2^{x_2} \ldots 3^{x_{12}}$$

and similarly

$$\Rightarrow f(M) = f(2^{y_1}3^{y_2} \ldots 37^{y_{12}} \ldots) = f(2)^{y_1} f(3)^{y_2} \ldots f(37)^{y_{12}} \ldots = 37^{y_1}2^{y_2} \ldots 3^{y_{12}}$$

$$f(N) = f(M) \Rightarrow x_1 = y_1, x_2 = y_2, \ldots, x_{12} = y_{12} \Rightarrow N = M$$

Hence the function is one-one.

Analysis/Comment: This is probably the most formidable yet the most beautiful question in the paper. The equation given in the question is an example of what is called as a functional equation. Such equations also find places in **JEE** exams occasionally but sometimes with an additional condition of being differentiable. The good thing about being differentiable is that there are only limited functions satisfying the condition and there are standard ways to identify them. That is however not the case here. In fact we cannot even talk about continuity of the function leave alone differentiability as the function is defined on a discreet set. There are infinite functions satisfying the given condition. The key here is therefore to come up with some suitable obvious function and then proving it indeed gives the least value. Such kind of hunch for an answer plays an important role in problem solving. It may be easy to identify the function, the challenge lies in justifying it serves the purpose. This is in sharp contrast to *Difficult to identify easy to prove* kinds of question like Q. 7. Justification however may be (and should be if it seems lengthy) skipped in an exam like this. But as in Q.7 & 8 here, one should be ready to justify the same. Also observe that the function given in the solution is not the only function giving the least value. Consider a variation of the above function with additional conditions $f(5) = 7$ and $f(7) = 5$. This will also give us the least required value and like this variation, one can come up with infinitely many variations.

Q.19 Let $x_1, x_2, \dots, x_{2014}$ be real numbers different from 1, such that $x_1 + x_2 + \cdots + x_{2014} = 1$ and

$$\frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \cdots + \frac{x_{2014}}{1-x_{2014}} = 1$$

What is the value of

$$\frac{x_1^2}{1-x_1} + \frac{x_2^2}{1-x_2} + \cdots + \frac{x_{2014}^2}{1-x_{2014}} = 1?$$

Ans: **0**

Hint: Manipulate the expression in terms of the given expressions.

Solution:

$$\sum \frac{x_1^2}{1-x_1} = \sum \frac{x_1 + x_1^2 - x_1}{1-x_1} = \sum \frac{x_1}{1-x_1} - \sum x_1 = 0$$

Analysis/Comment: The question is a good example of manipulation of a given expression. Infact the hint that one has to manipulate lies in the question itself. The question gives symmetric/cyclic expressions and asks the value of a symmetric/cyclic expression. Finding the values of $x_1, x_2, \dots$ is out of question. Such manipulations pay off in many problems especially in inequalities. There are countless examples where solution to an inequality is just one or couple of manipulations away. One must be good in doing the same. Such skills can be developed by mastering a handful of useful identities, using convenient notations, exploiting if the expression is cyclic/symmetric, making it a habit of solving a few questions mentally (A very useful habit, At least as many as half the questions in this paper are doable mentally) etc.

Q.20 What is the number of ordered pairs (A, B) where A and B are subsets of $\{1,2,\dots,5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$?

Ans: **570**

Hint: Use principle of Inclusion-Exclusion.

Solution: Let X denote the set of all ordered pairs (A, B) when $A \subseteq B$. Similarly let Y denote set of all ordered pairs (A, B) when $B \subseteq A$. The question asks to find $n(X' \cap Y') = n(S) - n(X \cup Y) = n(S) - n(X) - n(Y) + n(X \cap Y) = 2^{10} - 3^5 - 3^5 + 2^5 = 570$.

Analysis/Comment: The question is a good application of principle of inclusion and exclusion. Anyone not familiar with this, may start with the following problem in order to understand the principle.

Count the number of numbers from 1 *to* 1000 *which are either multiples of* 2 *or* 3. *Also count the numbers which are either multiples of* 2 *or* 3 *or* 5. *Try to generalize this and try using sets to generalize the same.*

Problems based on this principle are commonplace in competitive exams. A lots of problems based on finding the number of subsets/pairs of subsets of a set satisfying a given criteria can be tackled by using Venn Diagrams and this principle. It is advised to lookout for such questions in past exams.

Overall Comments:

As in the last year, the paper lives up to the expectations, infact does better by not having any bonuses this time around (although there was a correction in Q.17). All the important topics have been given fair share. There's however some duplication which could've been avoided. Q.3 and Q. 15 were direct applications of Pythagoras's theorem whereas Q.6 and Q.17 had quadratics with integer roots. There were questions where a good guesswork without justification seemed like saving lots of precious time, example being Q.7, 18. This might have been intentional, but commendable. Just like last year, inequalities don't appear directly, but are required to complete the solution of the problems like Q. 4, 11,13,14,18. The best problems are Q. 11, 14, 18.

It might look interesting that every year, they come up with problems with the exam year in it. There are as many as five problems here, however with exception of Q. 2, there is nothing special in the number 2014, one may just replace it with any other suitable candidate. Although there's no doubt it creates some sort of curiosity in a student, there's nothing more.

Lastly, as commented last year, as far as the preparation of such an exam goes, there unfortunately seems hardly any options available. The only serious effort taken from the concerned authorities is after the INMO, i.e. the 2^{nd} stage (3^{rd} stage in case of Mumbai). Many gifted students might just miss this because of lack of preparation in the very early stage. Anyone who is aspiring for doing well in Math Olympiads, must start working from the schools itself, as during the Class XI/XII, a student is more focused on engineering entrance exam preparations. It is not at all an impossible task as the syllabus for Olympiads is School Maths till class X. A reader here is invited to explore the of pre-college mathematics by following some standard available but less popular literature.